DIAMAGNETISM OF CONDUCTORS MOVING IN A MAGNETIC FIELD

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Several theoretical [1–3] and experimental [4] studies have been made of the diamagnetic perturbations during expansion of a conducting material in a magnetic field. These studies have related either to superconducting media [1], or to a strong magnetic field which has a considerable effect on the motion of the medium [2], or to a weakly ionized media, in which the effects of field variation in the medium can be neglected [3]. In the following we examine the expansion of a substance with finite conductivity in a weak (having no effect on the motion of the medium) magnetic field with account for the effects of field attenuation within the expanding matter. This occurs in the diagnostics of the state of the matter of a spark at a laser focus on the basis of diamagnetic induction signals [4]. The relations obtained in the following appear to be applicable for estimating the diamagnetic properties of meteor trails.

The method of solution of this problem may be of some interest; therefore, in the following the solution is obtained by several techniques for different basic geometries.

1. Magnetic-field penetration into a medium may involve two possible mechanisms—wave and diffusive.

For nonrelativistic velocities of the medium, wave effects play a negligibly small role in attenuation of the constant external magnetic field. The time for the diffusion of a magnetic field to the distance a in a medium with conductivity σ is $t \sim \sigma a^2 c^{-2}$ (c is the speed of light). It will be shown later that the magnetic field is weakened by the currents induced by the displaced field in a conductor traveling with velocity v. In all cases examined below the currents flow through a thin layer of the medium, whose thickness $\delta \sim c^2/4\pi\sigma v$ is much less than the dimension r_0 of the region occupied by the conductor. The diffusion of the magnetic field to this distance takes place [5] in the time

$$t \sim \sigma\delta^2 c^{-2} \sim \delta / 4\pi v \ll r_0 / 4\pi v \sim \tau ,$$

where τ is the characteristic time defining the rate of displacement of the magnetic field.

This inequality makes it possible to neglect the effects of magnetic-field lag during change of the currents which give rise to the field (i. e., we can consider the magnetic field to be quasi stationary) and we consider only the Lorentz force as the factor causing the electric current, so that $\mathbf{j} = \sigma c^{-1}[\mathbf{v} \times \mathbf{H}]$ (i. e., we neglect the electric field, whose curl is proportional to the time derivative of the magnetic field).

2. Let us examine the motion of a plane wave in the magnetic field **H**. In this section we take as the coordinate system the right-hand trihedron of vectors [x, y, z]. The wave travels along the y-axis and its parameters are the velocity v(y) and the conductivity $\sigma(y)$ of the medium. The quantities v and σ are constant in the plane perpendicular to the y-axis.

We first note that if in the plane perpendicular to the y-axis and intersecting it at the point y_0 there flows a current in the positive direction of the x-axis with surface density J, then the magnetic field of this current equals $2\pi J c^{-1}$ and is directed in the negative direction of the z-axis for $y < y_0$ and in the positive direction for $y > y_0$.

Returning to the plane wave, we note that if the magnetic field is directed parallel to the velocity of the medium, then there is no interaction of the wave with the field. We therefore examine the case in which the magnetic field has the component H_0 perpendicular to the velocity of the medium. We assume that this component is directed in the positive direction of the z-axis. Currents will obviously flow along the x-axis and they are considered positive if they flow in the positive direction of the x-axis. The current induced at the point y is $j(y) = H_Z \sigma v/c$. In this case $H_Z = H_0 + H_1$, where H_1 is the field induced by the currents j. Considering the different direction of the field from the currents flowing in the $y' < y$ and $y' > y$ planes, we obtain

$$H_1 = \frac{2\pi}{c}\int_{-\infty}^{y} j(y')\,dy' - \frac{2\pi}{c}\int_{y}^{\infty} j(y')\,dy'$$

and, consequently, the equation for j(y) has the form

$$j(y) = \frac{\sigma(y)\,v(y)}{c}\left\{H_0 + \frac{2\pi}{c}\int_{-\infty}^{y} j(y')\,dy' - \frac{2\pi}{c}\int_{y}^{\infty} j(y')\,dy'\right\}.$$

Its solution is the function

$$j(y) = \frac{cH_0}{2\pi}\,w(y)\,\frac{e^{-W(y)}}{1 + e^{-W(-\infty)}},$$

$$w(y) \equiv \frac{4\pi\sigma(y)\,v(y)}{c^2}, \qquad W(y) = \int_{y}^{\infty} w(y')\,dy'.$$

In this case the magnetic field component perpendicular to the velocity is

$$H_z(y) = 2H_0\,\frac{e^{-W(y)}}{1 + e^{-W(-\infty)}}.$$

Let us examine the case in which the quantity σv, remaining positive, increases monotonically from very small values for large positive y up to large values for large negative y (so that $W(-\infty) \gg 1$). Then

$$j(y) \approx \frac{cH_0}{2\pi}\,w(y)\,e^{-W(y)}, \qquad H_z(y) \approx 2H_0 e^{-W(y)}.$$

For large negative y the quantities j(y) and $H_z(y)$ are exponentially small because of the factor $\exp[-W(y)]$. This is associated with the fact that the magnetic field is attenuated in the depth of the wave and currents are not excited. For large positive y the quantity j(y) is small because of the factor $\sigma v/c$, i. e., because of the small conductivity or velocity the currents cannot be very strong. For these same y the quantity $H_z \approx 2H_0$, which can be considered a consequence of the combination of the original and displaced magnetic fields. The currents reach maximum values at the point where

$$\frac{dw(y)}{dy} + w^2(y) = 0.$$

3. Similar results are obtained when examining a cylindrical wave in a magnetic field parallel to its axis. We first examine a cylindrical wave of infinite length.

It is well known that the field of an infinitely long cylindrical layer of radius r, thickness dr, in a flow with current density j(r) perpendicular to the generator is equal to zero outside this layer, while within the layer the field is parallel to the cylinder axis and equal to $dH = 4\pi c^{-1} j(r)dr$.

If the layer has the conductivity σ and the currents are induced as it expands with the velocity v in the external magnetic field H, then $j = \sigma v c^{-1} H$. The magnetic field of this current within the cylinder will be opposite to the direction of the external field. Therefore in the cylindrical expansion wave the equation for the currents j(r) has the form

$$j(r) = \frac{\sigma(r)\,v(r)}{c}\left\{H_0 - \frac{4\pi}{c}\int_{r}^{\infty} j(r')\,dr'\right\}. \qquad (3.1)$$

Its solution is

$$j(r) = \frac{cH_0}{4\pi}\,w(r)\exp[-W(r)].$$

The magnetic field in this motion is

$$H(r) = H_0 \exp[-W(r)] .$$

This example is interesting in that in the case of finite radius of the wave ($\sigma v = 0$ for $r = r_0$) the magnetic field outside the wave does not change, although it is displaced within the wave. However, this is not so if the wave has large (in comparison with the radius) but finite length l. Such a cylinder creates at large (in comparison with the dimensions) distances a magnetic field which is the same as that of a magnetic dipole with the moment

$$M_{eff} \approx \frac{\pi l}{c} \int_0^{r_0} r^2 j(r)\, dr . \tag{3.2}$$

If the quantity σv grows sufficiently rapidly with reduction of r, then the function j(r) has a sharp maximum at the point r^*, at which

$$\frac{dw}{dr} + w^2 = 0 ,$$

and is exponentially small for small r. Then

$$\int_0^{r_0} r^2 j(r)\, dr \approx r^{*2} \int_0^{r_0} j(r)\, dr .$$

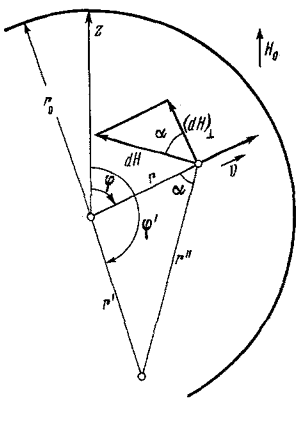
From (3.1), taken at the point r = 0, where j(r) = 0, we obtain

$$\int_0^{r_0} j(r)\, dr = \frac{cH_0}{4\pi} .$$

Consequently

$$M_{eff} \approx {}^1/_4 H_0 l r^{*2}. \tag{3.3}$$

4. A somewhat more complicated result is obtained in the case in which the magnetic field is directed perpendicular to the axis of a cylindrical expansion wave. Let us examine the case in which the wave has the sharp boundary $r_0(t)$, the velocity v of the medium is directed along the radius and, like the conductivity σ, depends only on the distance to the axis of the cylinder. In the coordinate system adopted in section 3, we consider the external magnetic field H_0 to be directed along the z-axis in the positive direction, the cylinder axis is directed along the x-axis; the currents are considered positive if they flow in the positive direction of the x-axis. We introduce the polar coordinates r and $\varphi (0 \le \varphi \le 2\pi)$ in the zy-plane (figure).



The magnetic field component inducing the current is

$$H_\varphi = -H_0 \sin \varphi .$$

The magnetic field at the point (r, φ) owing to the currents flowing through the point (r', φ') is (the notations are clear from the figure)

$$dH_1 = \frac{2}{c} j(r', \varphi') \frac{r'}{r''} dr' d\varphi'$$

$$(dH_1)_\varphi = -\frac{2}{c} j(r', \varphi') \frac{r'}{r''} \cos\alpha \, dr' \, d\varphi', \quad \cos\alpha = \frac{r - r'\cos(\varphi' - \varphi)}{r''} .$$

Thus

$$H_\varphi(r, \varphi) = -H_0 \sin\varphi - \frac{2}{c} \int_0^{r_0} dr' \int_0^{2\pi} d\varphi' j(r', \varphi') \frac{r - r'\cos(\varphi' - \varphi)}{r^2 + r'^2 - 2rr'\cos(\varphi - \varphi')} ,$$

$$j(r, \varphi) = -\sigma(r) v(r) c^{-1} H_\varphi(r, \varphi) . \quad (4.1)$$

If we seek the solution of (4.1) in the form

$$j(r, \varphi) = j(r) \sin\psi ,$$

then the equation for the function $j(r)$ has the form

$$j(r) = \frac{\sigma(r) v(r)}{c} \left\{ H_0 + \frac{2\pi}{cr^2} \int_0^r j(r') r'^2 dr' - \frac{2\pi}{c} \int_r^{r_0} j(r') dr' \right\} . \quad (4.2)$$

For an arbitrary dependence of the function σv on the coordinates this equation cannot be solved in general form. An approximate solution of this equation is obtained similarly to that presented in the following section for the most interesting practical case of a spherical expansion wave in a magnetic field.

5. Let us examine a sphere of radius $r_0(t)$, expanding in the magnetic field H_0. The velocity v of the medium is directed along the radius and, like the conductivity σ, depends only on the distance to the center of the sphere. We introduce the spherical coordinate system (r, ϑ, φ) with origin at the center of the sphere and direct the z-axis along the magnetic field. It is shown in [6] that if a current with surface density $i_0 \sin\vartheta$ (the current is considered positive if it flows clockwise) flows through a sphere of radius R along the lines ϑ = const the magnetic field of such a sphere has the components

$$H_r = -\frac{8\pi}{3} \frac{i_0 \cos\vartheta}{c} \begin{cases} 1 , & r < R \\ (R/r)^3, & r > R \end{cases} ,$$

$$H_\vartheta = \frac{8\pi}{3} \frac{i_0 \sin\vartheta}{c} \begin{cases} 1 , & r < R \\ -\frac{1}{2}(R/r)^3, & r > R \end{cases} .$$

Returning to the examination of the expanding sphere, we first note that since the velocity is radial the current gives rise only to the H_ϑ component, and if $j(r, \vartheta) \sim \sin\vartheta$ then $H_\vartheta \sim \sin\vartheta$ as well, i. e., we can seek the solution of the problem of current excitation in such a sphere in the form $j(r, \vartheta) = j(r) \sin\vartheta$.

If we examine the current at the distance r from the center of the sphere, then all the currents flowing in the layers inside the sphere yield a contribution to H_ϑ:

$$H_\vartheta^{(1)} = -\frac{4\pi \sin\vartheta}{3cr^3} \int_0^r r'^3 j(r') dr' .$$

The external layers yield

$$H_\vartheta^{(2)} = \frac{8\pi \sin\vartheta}{3c} \int_r^{r_0} j(r') dr' .$$

Consequently

$$H_{\vartheta} = -\sin\vartheta \left\{ H_0 + \frac{4\pi}{3cr^3} \int_0^r j(r)\, r^3 dr - \frac{8\pi}{3c} \int_r^{r_0} j(r)\, dr \right\}.$$

Since

$$j(r, \vartheta) = -\sigma(r)\, v(r) c^{-1} H_{\vartheta},$$

the equation for j(r) has the form

$$j(r) = \frac{\sigma(r)\, v(r)}{c} \left\{ H_0 + \frac{4\pi}{3cr^3} \int_0^r j(r)\, r^3 dr - \frac{8\pi}{3c} \int_r^r j(r)\, dr \right\}. \tag{5.1}$$

Just as in the cylindrical case, (5.1) cannot be solved for an arbitrary dependence of σ and v on r.

By using the formulas given in [6] we can write the potential of the magnetic field outside the sphere in the form

$$A_{\varphi} = \frac{4}{3} \frac{\pi \sin\vartheta}{cr^2} \int_0^{r_0} j(r)\, r^3 dr.$$

Comparing this expression with the formula for the field of a magnetic dipole with the moment $A_{\varphi} = Mr^{-2} \sin\vartheta$, we find that the effective magnetic moment of the expanding sphere is

$$M_{eff} = \frac{4}{3} \frac{\pi}{c} \int_0^{r_0} j(r)\, r^3 dr. \tag{5.2}$$

The absence of an exact solution for j(r) prevents exact calculation of M_{eff}; however, in certain cases it is possible to obtain approximate estimates of this quantity.

a) Small velocities and conductivities. The integral equation (5.1) for j(r) can be compared with the differential equation

$$\frac{d}{dr}\left[r^4 \left(\frac{d}{dr} \frac{j(r)}{w(r)} - j(r) \right) \right] + r^3 j(r) = 0. \tag{5.3}$$

If $w(r)r \ll 1$ for all $r(0 < r < r_0)$, then by neglecting small terms in (5.3) we obtain

$$\frac{d}{dr}\left[r^4 \frac{d}{dr} \frac{j(r)}{w(r)} \right] = 0. \tag{5.4}$$

The solution of (5.4) is the function

$$j(r) = w(r)\, (C_1 + C_2 r^{-3}).$$

For $r = r_0$ we obtain from (5.1)

$$j(r_0) = w(r_0) \left\{ \frac{cH_0}{4\pi} + \frac{1}{3r_0^3} \int_0^{r_0} r^3 j(r)\, dr \right\}, \quad \left[\frac{d}{dr} \frac{j(r)}{w(r)} \right]_{r=r_0} = j(r_0) - \frac{1}{r_0^4} \int_0^{r_0} r^3 j(r)\, dr.$$

From these equations we find

$$C_1 = \frac{H_0 c}{4\pi}, \quad C_2 = \frac{H_0 c}{8\pi} \int_0^{r_0} r^3 w(r)\, dr \ll C_1 r_0^3.$$

Consequently

$$j(r) \approx \frac{H_0 c}{4\pi} w(r), \quad M_{eff} \approx \frac{4\pi H_0}{3c^2} \int_0^{r_0} \sigma(r)\, v(r)\, r^3 dr \,, \tag{5.5}$$

which coincides exactly with the result of [3].

b) Large velocities and conductivities. If the quantity w grows rapidly with reduction of r, so that

$$rw(r) \gg 1$$

for $r < r_0$, then by analogy with the plane case we can conclude that the current flows only in a narrow region and decreases exponentially with reduction of r. In this case we can write

$$j(r) = f(r)\, w(r) \exp\left[-\alpha \int_r^{r_0} w(r)\, dr\right] \,,$$

where the factor $\alpha \sim 1$ and the slowly varying function $f(r)$ appear in connection with the possible influence of the geometry. Consequently, the current as a function of r has a maximum near the point r^*, where

$$\frac{dw}{dr} + \alpha w^2 = 0 \,.$$

If this maximum is sufficiently narrow, then the integral in (5.5), defining the magnetic moment, can be replaced by

$$r^{*3} \int_0^{r_0} j(r)\, dr \,.$$

Since in the case in question the function j(r) is exponentially small for small r, we obtain from (5.1), taken at the point r = 0,

$$0 \approx \frac{\sigma(0)\, v(0)}{c} \left\{H_0 - \frac{8\pi}{3c} \int_0^{r_0} j(r)\, dr\right\}, \quad \text{or} \quad \int_0^{r_0} j(r)\, dr \approx \frac{3}{8\pi} H_0 c \,,$$

so that

$$M_{eff} = {}^1\!/_2\, H_0 r^{*3} \,. \tag{5.6}$$

This formula is valid under two conditions:

1) the value of r^* depends weakly on the unknown α;

2) the width of the peak of the function j(r) is much less than r^*.

Satisfaction of the first condition depends on the concrete form of the function w(r). To define the second condition we write

$$w(r) \exp\left\{-\alpha \int_r^{r_0} w(r)\, dr\right\}$$

near r^* in the form

$$w(r^*) \exp\left\{-\alpha \int_{r^*}^{r_0} w(r)\, dr\right\} \exp\left\{-\frac{(r - r^*)^2}{2\delta^2}\right\} \,,$$

where the peak half-width δ must be considerably less than r^*, i.e.,

$$\delta \equiv \left[\left(\frac{w'}{w}\right)^2 - \frac{w''}{w} - \alpha w'\right]^{-1/2}_{r=r^*} \ll r^* ; \qquad (5.7)$$

for $r = r^*$ the ratio $w'/w^2 = -\alpha$, and consequently (5.7) can be written in the form

$$r^* \left[-2\alpha w'(r^*) - \frac{w''(r^*)}{w(r^*)}\right]^{1/2} \gg 1 .$$

c) In the case in which the quantity w is sufficiently large so that even at the boundary of the sphere $w' + \alpha w > 0$, then the current density decreases monotonically from the boundary into the depth of the sphere, and in this case we must replace r^* by r_0 in the formulas of case b). In this case we obtain the known formula defining the magnetic moment of an ideally conducting sphere expanding in a magnetic field [1].

6. The method developed above was used to compare the theoretical calculations with the experimental data of [4]. In this case the hydrodynamic and thermodynamic characteristics of the expanding medium were calculated on the basis of the blast model [7]. The magnitude of the conductivity was calculated using the formula*

$$\sigma = 1.05 \cdot 10^{16} T^\circ \left(\frac{\rho}{\rho_0}\right)^{-1/2} \exp\left\{-\frac{6.04 \cdot 10^4}{T^\circ}\right\} . \qquad (6.1)$$

Here T° is the air temperature in °K, and ρ_0 is the air density at standard conditions.

Since the initial energy release volume has an elongated form, whose longitudinal dimension ($l \sim 0.1$ cm) is considerably larger than the transverse dimension, and since the magnetic field is longitudinal, in this case we must use the results of section 3.

It was found that up to $t \sim 10^{-7}$ sec the expanding medium can be considered infinitely conducting. The magnetic moment can be calculated using (3.3), in which r^* should be replaced by r_0—the radius of the shock wave front.

At later times the radius of the region through which the currents flow becomes less than r_0. We can assume that it is at just these times that the magnetic moment growth stops. In the case in which this assumption is valid the maximum magnetic moment M_{max} is also defined by (3.3), in which r^* is now replaced by r_1, defined by the condition

$$[r_1 w(r_1)]_{r_1 = r_0} = 1 . \qquad (6.2)$$

If at these times the state of the gas at the shock wave front is still described by the Sedov solution, then (6.2) can be written in the form (here and hereafter all quantities are in the CGSE system)

$$0.9 \cdot 10^{-7}\, r_1 z^{3/4} \exp\{-5 \cdot 10^{12}/z\} = 1, \quad z = E/\rho_0 r_1^2 .$$

Neglecting the slowly varying preexponential factor, we obtain the solution of (6.2): $z \sim 5 \cdot 10^{12}$ or $r_1^2 = E\rho^{-1}10^{-12}/5$. Consequently

$$M_{max} = {}^1/_4 H_0 l r_1^2 = 5 \cdot 10^{-14} H_0 E l \rho_0^{-1} .$$

Substituting the numerical values of the quantities, we obtain for the case examined in [4]

$$M_{max} \sim 5 \cdot 10^{-5} H_0 .$$

It is not possible to give a more exact estimate of M_{max}, since the state of the air in the region behind the shock wave front is not described by the blast model of Sedov (for example, even a rough account for heat conduction alters significantly [7] the temperature distribution pattern in the internal regions).

The experimentally measured value of $M_{max} \sim 10^{-5} H_0$. A possible reason for this discrepancy is that not all the laser flash energy is released in the luminous gas breakdown spark. It appears that a considerable part of the energy is either scattered or passes through the spark region (in this connection we note that the estimate given above is obviously low, since increase of the magnetic moment clearly continues some time after separation of the currents

*Private communication of E. V. Pletnikov. His formula (6.1) describing the conductivity of air in the temperature range $(5-20) \cdot 10^3$ °K agrees with the experimental data presented in [8].

from the shock wave front).

The nature of the dependence of M_{max} on the energy released in the spark and the gas density coincides with the experimental relation.

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